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Strongly Unique Best Approximation in Uniformly Convex Banach Spaces

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Let Y be a closed subspace of a Banach X. An element z in Y is called a *best approximation* to an element x in X if

$$\|x-z\| = \inf_{y \in Y} \|x-y\|.$$

We say z is *strongly unique* at x, if there exists a $\gamma = \gamma(x) > 0$ such that, for all $y \in Y$,

$$\|x-y\| \geqslant \|x-z\| + \gamma \, \|y-z\|.$$

We say z is *strongly unique of order* α ($\alpha > 1$) at x, if, for some $M > 0$, there exists a $\gamma = \gamma(x, M) > 0$ such that, for all $y \in Y$ with $\|y-z\| \leqslant M$,

$$\|y-x \geqslant \|z-x\| + \gamma \, \|y-z\|^{\alpha}.$$

Let α be a strictly increasing function from R^+ into R^+ with $\alpha(0)=0$. z is called an *α-strongly unique best approximation* to x if there exists a positive number K such that

$$\alpha(\|x-y\|) \geqslant \alpha(\|x-z\|) + K\alpha(\|y-z\|)$$

for all y in Y. It is known [N – S] that if Y is a Haar subspace of $C(B)$, the space of continuous real-valued functions of a compact Hausdorff space B with the supremum norm, then for every x in $C(B)$ there exists a strongly unique best approximation in Y. Also it is well known that in a Hilbert space, every best approximation is an α-strongly unique best approximation with $\alpha(s) = s^2$ and $K = 1$. On the other hand, it is known [W] that a strongly unique best approximation need not exist for every x in X when X is smooth. Recently, J. R. Angelos, A. Egger, and R. Smarzewski have proved the following theorems.

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THEOREM A [A–E]. *Let Y be a finite dimensional subspace of L_p, $1<p<\infty$. Then the best approximation to $x \in L_p$ from Y is strongly unique of order* $\max(p, 2)$.

THEOREM B [Sm]. *Let Y be a subspace of L_p, $1<p<\infty$. Then the best approximation to $x \in L_p$ from Y is α-strongly unique with $\alpha(s)=s^{\max(p,2)}$.*

The main result in this paper is to extend the above theorems to uniformly convex spaces.

Recall a Banach space X is said to be *uniformly convex* if $\delta(\varepsilon)>0$ for $0<\varepsilon \leqslant 2$, where

$$\delta(\varepsilon)=\inf\{1-\|x+y\|/2 \mid \|x\|=\|y\|=1, \|x-y\| \geqslant \varepsilon\}.$$

We say that a uniformly convex space X has *modulus of convexity of power type p* if for some $0<k<\infty$, $\delta(\infty)$, $\delta(\varepsilon) \geqslant k\varepsilon^p$. It is known [H] that L_p, $1<p<\infty$, has modulus of convexity of power type max(2, p). For more geometric properties of Banach spaces, we suggest the reader consult [D]. For recent results of strongly unique best approximation in $C(B)$ see [B, B–M, C, N, P].

THEOREM 1. *Let X be a uniformly convex Banach space of power type p and let Y be a subspace of X. Then the best approximation to $x \in X$ from Y is strongly unique of order p.*

Proof. Without loss of generality we may assume that 0 is the best approximation to $x \neq 0$ from Y. For any unit vector y in Y and $0<a \leqslant 2\|x\|$, we have

$$3\|x\| \geqslant \|x+ay\| \geqslant \|x+ay/2\| \geqslant \|x\|.$$

Hence,

$$\begin{aligned}\|x+ay/2\| &\leqslant (1-ka^p/\|x+ay\|^p)\ \|x+ay\| \\ &\leqslant (1-ka^p/3^p\|x\|^p)\|x+ay\|.\end{aligned}$$

We have

$$\begin{aligned}\|x+ay\| &\geqslant \|x+ay/2\|+ka^p/(3^p\|x\|^{p-1}) \\ &\geqslant \|x\|+ka^p/(3^p\|x\|^{p-1}).\end{aligned}$$

Hence, 0 is strongly unique of order p where $M=2\|x\|$ and $\gamma=k/(3^p\|x\|^{p-1})$. ∎

THEOREM 2. *Let X be a uniformly convex Banach space of power type p* *and let Y be a subspace of X. Then the best approximation to $x \in X$ from Y is α-strongly unique with $\alpha(s) = s^p$.*

Proof. Without loss of generality, we may assume that 0 is the best approximation to $x \neq 0$ from Y. For any unit vector $y \in Y$ and $0 < a \leqslant 2\,\|x\|$, we have

$$\begin{aligned} \|x + ay\| &\geqslant \|x\| + ka^p/(3^p\,\|x\|^{p-1}) \\ &= \|x\|\,(1 + ka^p/(3^p\,\|x\|^p)). \end{aligned}$$

Hence, if $0 < a \leqslant 2\|x\|$, then

$$\begin{aligned} \|x + ay\|^p &\geqslant \|x\|^p\,(1 + ka^p/(3^p\,\|x\|^p))^p \\ &\geqslant \|x\|^p\,(1 + pka^p/(3^p\,\|x\|^p)) \\ &= \|x\|^p + pka^p/3^p. \end{aligned}$$

If $2\,\|x\| \leqslant a \leqslant 4\,\|x\|$, then

$$\begin{aligned} \|x + ay\|^p &\geqslant \|x + 2\|x\|\,y\|^p \\ &\geqslant \|x\|^p\,(1 + pk(2\,\|x\|)^p/(3^p\,\|x\|^p)) \\ &= \|x\|^p\,(1 + pk(\tfrac{2}{3})^p) \\ &= \|x\|^p + pk(\tfrac{1}{6})^p\,(4\,\|x\|)^p \\ &\geqslant \|x\|^p + pk(\tfrac{1}{6})^p\,a^p. \end{aligned}$$

If $a > 4\,\|x\|$, then

$$\begin{aligned} \|x + ay\|^p &\geqslant |a - \|x\|\,|^p \\ &\geqslant \|x\|^p + (a - 2\,\|x\|)^p \\ &\geqslant \|x\|^p + (a/2)^p. \end{aligned}$$

So 0 is α-strongly unique $K = \min((\tfrac{1}{2})^p, pk(\tfrac{1}{6})^p)$ and $\alpha(s) = s^p$. ∎

Remark 1. Theorem 1 and Theorem 2 are still true if one replaced the subspace Y by a nonempty closed convex subset M (see [Sm2, P–Sm].

Remark 2. Theorem 2 is also proved by B. Prus and R. Smarzewski [Sm2, P-Sm]. The above proof is much simpler than their proof.

Recall that a Banach space X is said to be *uniformly mooth* if

$$\lim_{\tau \to 0^+} \rho(\tau)/\tau = 0,$$

where

$$\rho(\tau) = \sup\{(\|x+y\| + \|x-y\|)/2 - 1 \mid \|x\| = 1, \|y\| = \tau\}.$$

We say that a uniformly smooth space X has *modules of smoothness of power type p* if for some $0<p<\infty$, $\rho(\tau) \leqslant k\tau^p$. It is known that L_p, $1<p<\infty$, has modulus of smoothness of power type $\min(2, p)$.

THEORM 3. *Let X be a uniformly smooth Banach space of power type p and let $Y \neq \{0\}$ be a subspace of X. Then the best approximation to $x \in X \setminus Y$ from Y cannot be strongly unique of order q for $q<p$.*

Proof. Without loss of generality, we may assume that 0 is the best approximation to $x \neq 0$ from Y. For any unit vector $y \in Y$, there is $k>0$ such that for all $a>0$

$$(\|x+ay\| + \|x-ay\|)/2 - \|x\| \leqslant k(a/\|x\|)^p \cdot \|x\|.$$

Hence, for fixed $a>0$, either

$$\|x+ay\| \leqslant \|x\| + k(a/\|x\|)^p \|x\|$$

or

$$\|x-ay\| \leqslant \|x\| + k(a/\|x\|)^p \|x\|.$$

So 0 is not strongly unique of order q for $q<p$. ∎

Similarly, we have the following theorem.

THEOREM 4. *Let X be a uniformly smooth Banach space of power type p and let $Y \neq \{0\}$ be a subspace of X. Then the best approximation to $x \in X \setminus Y$ from Y cannot be α-strongly unique with $\alpha(s) = s^q$ for $q<p$.*

Proof. Without loss of generality, we may assume 0 is a best approximation of $x \neq 0$ from Y. For any unit vector $y \in Y$ and $a>0$ we have either

$$\|x+ay\| \leqslant \|x\| + k(a/\|x\|)^p \|x\|$$

or

$$\|x-ay\| \leqslant \|x\| + k(a/\|x\|)^p \|x\|.$$

Since there exist $\delta>0$ and $k'>0$ such that if $0<b<\delta$ then

$$(1+b)^p \leqslant 1+k'b,$$

we have

$$\min(\|x+ay\|^p, \|x-ay\|^p, \|x-ay\|^p) \leqslant \|x\|^p + k'ka^p$$

whenever $0 < a^p k < \delta \|x\|^p$. So 0 is not α-strongly unique with $\alpha(s) = s^q$ for $q < p$. ∎

Remark 3. We say that a metric projection P is *pointwise Lipschitz continuous* of order r $(0 < r \leqslant 1)$ at x if there exist $M > 0$ and $\lambda > 0$ such that

$$\|Px - Pz\| \leqslant \lambda \|x - z\|^r$$

whenever $\|z - x \leqslant M$. B. L. Chalmers and G. D. Taylor [Ch–T] proved that the metric projection P is pointwise Lipschitz continuous of order $1/\alpha$ at x, whenever Px is strongly unique of order α at x. Also, Björnestal [Bj] proved that if X is q-uniformly convex and p-uniformly smooth, then P is pointwise Lipschitz continuous of order p/q from X into Y (or a closed convex subset M). Here, we give a simple proof.

Since X is q-uniformly convex, Px is strongly unique of order q to x from Y. So

$$\|x - (Px + Pz)/2\| \geqslant \|x - Px\| + \gamma \|Px - Pz\|^q/2^q$$

or

$$\gamma \|Px - Pz\|^q/2^q \leqslant \|x - (Px + Pz)/2\| - \|x - Px\|$$

for any $\|z\| \leqslant M$. (Since $\|\cdot\|$ is a convex function, the definitions of "strongly unique of order α" and "pointwise Lipschitz continuous of order r" in our paper are equivalent to definitions in p. 34 and p. 39 in [Ch–T]). Similarly,

$$\gamma' \|(Px - Pz)/2\|^q < \|z - (Px + Pz)/2\| - \|z - Pz\|.$$

Hence,

$$\begin{aligned}
\gamma \|Px - Pz\|^q/2^q
&\leqslant (\gamma + \gamma') \|Px - Pz\|^q/2^q \\
&\leqslant \|z - (Px + Pz)/2\| + \|x - (Px + Pz)/2\| \\
&\quad - \|x - Px\| - \|z - Pz\| \\
&\leqslant \|z - (Px + Pz)/2\| + \|x - (Px + Pz)/2\| \\
&\quad - \|x + z - (Px + Pz)\| \\
&\leqslant k \|x - z\|^p \cdot \|x + z - (Px + Pz)/2\|^{1-p}.
\end{aligned}$$

So if $\|x-z\| < \|x-Px\|/4$, then

$$\|Px - Pz\| \leqslant k' \|x-z\|^{p/q} \qquad \text{for some } k'.$$

For some other results, see [Ab].

Remark 4. It is known that if $c^p > a^p + b$ and $0 < b < M$, then there is $k > 0$ so that

$$c > a + kb.$$

Hence, α-strong uniqueness with $\alpha(s) = s^p$ implies strong uniqueness of order p. (By the proof of Theorem 2, these two definitions are equivalent.)

Remark 5. It is known that if X is uniformly convex (or uniformly smooth) then X is reflexive (see [L–T, p. 61, Proposition l.e.3]). It is also known that X has modulus of convexity of power type p if and only if X^* has modulus of smootness of power type $p/(p-1)$ (see [L–T, p. 63]).

Remark 6. T. Figiel [F] has computed the modulus of convexity of $L_p(X)$, $1 < p < \infty$, the spaces of all measurable X-valued functions f on $[0, 1]$ such that

$$\|f\|_{L_p(X)} = \left(\int \|f(t)\|_X^p \, dt \right)^{1/p} < \infty.$$

He proved that if X has modulus of convexity of power type q, then $L_p(X)$ has modulus of convexity of power type $\max(p, q)$. So $L_p(L_q)$ has modulus of convexity (smoothness) of power type $\max(p, q, 2)$ ($\min(p, q, 2)$). It is known that $L_p(L_q)$ is not an L_p-space unless $p = q$.

Remark 7. The Schatten space C_p, $1 \leqslant p < \infty$, is the set of all compact operators x on l_2 for which $\|x\|_p = (\text{trace}(x^*x)^{p/2}))^{1/p}$. It is known that if $p \neq 2$, then C_p is not a Banach lattice [Lw]. N. Tomczak-Jaegermann [T] has shown that for $1 < p < \infty$, the C_p classes have moduli that behave like the L_p-spaces. So C_p has modulus of convexity (smoothnes) or power type $\max(p, 2)$ ($\min(p, 2)$).

Remark 8. G. Pisier [Pi] has shown that every superflexive Banach space (for definition see [D, p. 86]) can be renormed as a space with modulus of convexity (smoothness) of power type p for some $p < \infty$ (q for some $q > 1$). Note: every p-convex Banach lattice is super-reflexive [L–T, Theorem 1.f.1, p. 80].

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